6

Weak Measurements, the Energy-Momentum Tensor and the Bohm Approach

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Abstract

In this paper we show how the weak values, $\langle \boldsymbol{x}(t)|\widehat{P^{\mu}}|\psi(t_0)\rangle/\langle \boldsymbol{x}(t)|\psi(t_0)\rangle$, are related to the $T^{0\mu}(\boldsymbol{x},t)$ component of the energy-momentum tensor. This enables the local energy and momentum to be measured using weak measurement techniques. We also show how the Bohm energy and momentum are related to $T^{0\mu}(\boldsymbol{x},t)$ and therefore it follows that these quantities can also be measured using the same methods. Thus the Bohm 'trajectories' can be empirically determined as was shown by Kocis *et al* in the case of photons. Because of the difficulties with the notion of a photon trajectory, we argue the case for determining experimentally similar trajectories for atoms where a trajectory does not cause these particular difficulties.

6.1 Introduction

The notion of weak measurements introduced by Aharonov, Albert and Vaidman (1988,1990) has opened up a radically new way of exploring quantum phenomena. In contrast to the strong measurement (von Neumann 1955), which involves the collapse of the wave function, a weak measurement induces a more subtle phase change which does not involve any collapse. This phase change can then be amplified and revealed in a subsequent strong measurement of a complementary operator that does not commute with the operator being measured. This amplification explains why it is possible for the result of a weak spin measurement of a spin-1/2 atom to be magnified by a factor of 100 (Aharonov *et al.* 1988, Duck, Stevenson and Sudarshan 1989). A weak measurement, then, provides a means of amplifying small signals as well as allowing us to gain new, more subtle information about quantum systems.

One of the new features that we will concentrate on in this paper is the possi-

ble measurement of the $T^{0\mu}(\boldsymbol{x}, t)$ components of the energy-momentum tensor. In section 6.3.2 we will show that these components are related to the real part of the weak value $\langle P^{\mu}(t)\rangle_W$ through the expression

$$\Re\langle P^{\mu}(t)\rangle_W = T^{0\mu}(\boldsymbol{x}, t), \tag{6.1}$$

where the weak value is defined by

$$\langle P^{\mu}(t)\rangle_W := \frac{\langle \boldsymbol{x}(t)|\widehat{P^{\mu}}|\psi(t_0)\rangle}{\langle \boldsymbol{x}(t)|\psi(t_0)\rangle}.$$

This allows us, not only to discuss, but also to actually measure, the *local* properties of the energy, $E(\boldsymbol{x}, t)$, and the momentum, $P(\boldsymbol{x}, t)$. These results are in contrast with the standard treatments, which only discuss the *global* properties, using the expressions

$$E(t) = \int T^{00}(\boldsymbol{x}, t)d^3x \qquad \text{and} \qquad P^j(t) = \int T^{0j}(\boldsymbol{x}, t)d^3x.$$

The use of these global quantities are usually justified by claiming that in quantum mechanics, it is only through these global quantities that energy and momentum can be uniquely defined and conserved (Schweber, 1961)

However we can show that the *local* energy and momentum can also be conserved. The way to do this comes from a surprising direction – the Bohm approach (Bohm 1952). Critically examining the mathematical structure in the simple case of the non-relativistic, spin-zero particle, we find the real part of the Schrödinger equation under polar decomposition gives the quantum Hamilton-Jacobi equation [qHJ][1], namely,

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + Q + V = 0 \tag{6.2}$$

where Q is a novel form of energy, which we call the quantum potential energy. This takes the form

$$Q(\boldsymbol{x}, t) = -\frac{\nabla^2 R(\boldsymbol{x}, t)}{2mR(\boldsymbol{x}, t)}.$$

If we now follow Bohm and define

$$P_B(\boldsymbol{x}, t) := \nabla S(\boldsymbol{x}, t) \qquad \text{and} \qquad E_B(\boldsymbol{x}, t) := -\partial_t S(\boldsymbol{x}, t). \tag{6.3}$$

We then find that the qHJ equation becomes a simple local energy conservation equation

$$E_B(\boldsymbol{x}, t) = (P_B(\boldsymbol{x}, t))^2/2m + Q(\boldsymbol{x}, t) + V(\boldsymbol{x}, t),$$

[1] We use $\hbar = 1$ throughout this paper.

provided we regard the quantum potential energy as a new quality of energy appearing only in the quantum domain. Indeed when $Q = 0$, we have the classical equation for the conservation of energy. In this case S becomes the classical action and we recapture the classical equations of motion.

Hiley and Callaghan (2010a, 2010b) have shown further that Bohm's conjectured expressions for the momentum and energy given by (6.3) can be put on a firmer footing by showing that

$$\rho P_B^j(\boldsymbol{x}, t) = T^{0j}(\boldsymbol{x}, t) \qquad \text{and} \qquad \rho E_B(\boldsymbol{x}, t) = T^{00}(\boldsymbol{x}, t) \tag{6.4}$$

where ρ is the probability density and where expressions for P_B^j and E_B are obtained from the expressions of the Langrangian for the Pauli and Dirac particles repsectively. A more detailed explanation of these results will be given in section 3.

Thus our results hold, not only for Schrödinger particles, but also for Pauli and Dirac particles. The corresponding real parts of the Pauli and Dirac equations give rise to their respective qHJ equations. Both these equations contain their corresponding expressions for the quantum potential energies (see Hiley and Calaghan 2010c). Thus the quantum potential energy, the existence of which was regarded as *ad hoc* by Heisenberg (1958) and unnecessary by Dürr *et al* (1996), plays a vital role in the conservation of local energy and is at the heart of quantum theory.

A comparison of equations (6.1) and (6.4) shows the relation between the weak values and the parameters introduced by Bohm, for example, the weak value $\langle P_x \rangle_W$ is exactly the x-component of the Bohm momentum. Indeed Leavens (2005), Wiseman (2012) and Hiley (2012) have already shown that this is a particular example of a more general result applicable to the Schrödinger particle so that analogous weak values can be found for the corresponding Bohm energy and Bohm kinetic energy. Similar relations also apply to relativistic particles with spin (Hiley and Callaghan, 2010c). This shows that these quantities are not arbitrarily added 'philosophical' terms, but actually correspond to entities that can be measured in the laboratory.

Thus the Bohm approach, introduced originally to show that it is possible to provide a realist model of quantum phenomena without the need for the observer to play an essential role in the theory, can no longer be criticised on the grounds that it uses unobservable terms like the Bohm momentum and Bohm energy. In some cases these quantities have been measured using weak measurement techniques (Kocsis *et al.* 2011). Furthermore the criticism that the notion of particles following unobserved 'trajectories' adds no new physical content to the theory can also no longer be sustained. Thus what appeared to be an empty physical theory actually adds new insight to standard quantum mechanics, and is not in opposition to it as is often perceived. It actually enriches the standard theory.

The recent experimental results of Kocis *et al* (2011), confirm that the usual

criticism can no longer hold. These authors empirically determine an ensemble of what they call 'photon trajectories' in regions of interference where standard arguments using the uncertainty principle would suggest that the interference must be destroyed (See Figure 6.1).

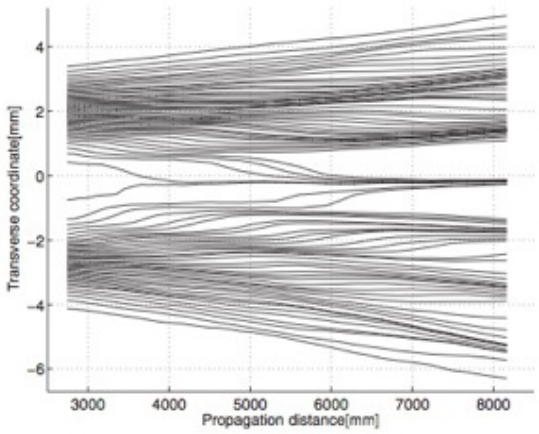

Figure 6.1 Experimentally produced photon 'trajectories' (See Kocsis *et al.* 2011.

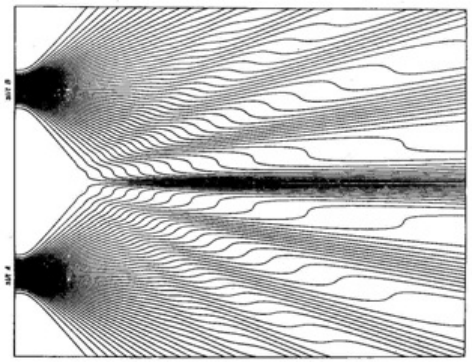

Figure 6.2 Theoretical Schrödinger particle trajectories (see Philippidis *et al.* 1979).

Although the notion of a trajectory for a photon is a not without its own difficulties, these photon 'trajectories' have a striking resemblance to the particle trajectories calculated in Philippidis, Dewdney and Hiley (1979) for a Schrödinger particle in the Bohm model (See Figure 6.2). The question we want to discuss in this paper is whether we can use these techniques for atoms rather than photons. In this case difficulties associated with photon trajectories do not arise. Using atoms in the

non-relativistic region will enable us to make a direct comparison with the theoretical calculations of Philippides *et al*, (1979). We will discuss our experimental proposals in Section 5.

6.2 Quantum Measurements.

6.2.1 von Neumann Measurements

The usual notion of a quantum measurement, the von Neumann measurement, involves a process in which the wave function 'collapses' into one of the eigenfunctions of the operator whose value is required. Let us recall how this works in some detail to enable us to directly contrast it with what is involved in a weak measurement.

Suppose we want to measure a property of our system which is described by an operator $\hat{A}$ such as, for example, the spin of a particle. One way to determine this is to pass the particle through an inhomogeneous Stern-Gerlach magnetic field which deflects the particles according to their spin state. If the particles are initially traveling along the y-axis, then we orient the field in transverse direction relative to the line of flight, say, along the z-axis. The field then separates the particles into their two spin states. The 'pointer reading' then corresponds to the position of the final two peaks, spin up, spin down.

With the Stern-Gerlach experiment, [SG], in mind, we follow Bohm (1951) and use the interaction Hamiltonian $H_I = g(t)\hat{Z}\,\hat{\sigma}_z$, so that the position operator $\hat{Z}$ is coupled to the spin operator $\hat{\sigma}_z$. $g(t)$ is some function describing the strength of the interaction. The problem then is to solve the Schrödinger equation using this Hamiltonian. We will remind readers of how this is done in a standard (strong) measurement simply to contrast it to a weak measurement.

We introduce the time development operator, $U(t, t_0)$, to determine the final ket $|\Psi(t)\rangle$ from some initial ket $|\Psi(t_0)\rangle$, so that

$$|\Psi(t)\rangle = U(t, t_0)|\Psi(t_0)\rangle.$$

$U(t, t_0)$ satisfies

$$\frac{\partial U(t, t_0)}{\partial t} = -iH(t)U(t, t_0)$$

which gives the solution

$$U(t, t_0) = \exp\left[-i\int_{t_0}^{t} H(t')dt'\right].$$

We now substitute the interaction Hamiltonian with the condition

$$\eta = \int_{t_0}^{t} g(t')dt' = D\Delta t$$

where D is a measure of the strength of the interaction and Δt is the time the interaction is active. This gives the final state as

$$|\Psi(t)\rangle = \exp\left[-i\eta\hat{Z}\,\hat{\sigma}_z\right]|\Psi(t_0)\rangle. \tag{6.5}$$

Then

$$\langle z|\Psi(t)\rangle = \int \langle z|\exp\left[-i\eta\hat{Z}\,\hat{\sigma}_z\right]|z'\rangle\langle z'|\Psi(t_0)\rangle dz'.$$

so that we have

$$\Psi(z,t) = \exp\left[-i\eta z\hat{\sigma}_z\right]\Psi(z,t_0).$$

Let us choose the total initial spin state to be

$$\Psi(z,t_0) = \psi(z,t_0)\sum_{n=1}^{2} c_n|\xi_n\rangle$$

where $|\xi_n\rangle$ are the eigenkets of $\hat{\sigma}_z$, viz, $\hat{\sigma}_z|\xi_n\rangle = a_n|\xi_n\rangle$ where, for a spin-half system, a_n take the values $+1$ or -1. The final state will be

$$\Psi(z,t) = c_+\psi(z-\eta) + c_-\psi(z+\eta). \tag{6.6}$$

Thus we are left with two sharply peaked wave packets, one centred at $\eta = D\Delta t$ and the other at $\eta = -D\Delta t$. In the case when the initial wave function is Gaussian, $|\psi(t_0)\rangle = \int \exp\left[-\frac{z'^2}{4(\Delta z)^2}\right]|z'\rangle dz'$, the final wave function in the momentum representation would take the form

$$\phi(p,t) = A_+\exp[-(\Delta z)^2(p^2-\eta)] + A_-\exp[-(\Delta z)^2(p^2+\eta)]. \tag{6.7}$$

If the interaction is sufficiently strong to ensure the two wave packets are well separated, the 'pointer' position can distinguish the two spin states (See Figure 6.3).

Each particle will end up in one of the wave packets and will be detected at the appropriate point on the screen. This means that at each detection, the wave function 'collapses' into one or other of the two packets described in equation (6.6), in which it then remains. It is this process that is called a von Neumann or strong measurement.

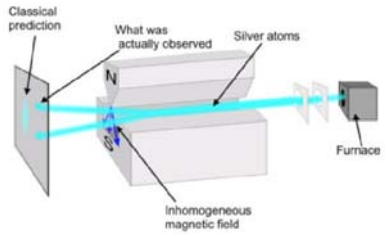

Figure 6.3 Diagram of the classic Stern-Gerlach experiment. A beam of silver atoms is produced in a furnace and passed through a very strong inhomogeneous magnetic field. The beam is split into its two spin components.

6.2.2 Example of Weak Measurements for Spin.

In order to make a weak measurement, we must reduce the strength of the field of the Stern-Gerlach magnet to the point where the two wave packets do not completely separate, but still overlap so that interference effects will be present. More precisely, if the incident wave packet is a Gaussian, the spatial part of the packets become separated in the z-direction. If the separation of their centres is, say, a distance a, and this is less than the width, Δz, of the incoming wave packet, the packets will interfere. Detailed calculations sketched below show that the resulting beam approximates to a Gaussian, but with its centre displaced by an amount that depends on the weak value $\langle\sigma_z\rangle_W = \langle\xi_f|\hat{\sigma}_z|\xi_i\rangle/\langle\xi_f|\xi_i\rangle$.

Further interesting results can be obtained if we introduce a further Stern-Gerlach magnet to make a strong measurement in, say, the x-direction. In this case the beam is split into two, this time along the x-direction. If these two beams are allowed to separate completely, then we see two Gaussians whose centres are displaced from two Gaussians that would have been produced if the weak Stern-Gerlach magnet had not been in place. The displacement of each packet enable the weak value $\langle\sigma_z\rangle_W = \langle\xi_f|\hat{\sigma}_z|\xi_i\rangle/\langle\xi_f|\xi_i\rangle$ to be measured.

Figure 6.4 illustrates how all of this comes about. A beam of neutral atoms, moving along the y-axis, produced from an oven, are first polarised at an angle α to the x-axis. The beam then passes through an inhomogeneous magnetic field oriented along the z-axis, $\mathrm{SG_z}$, which produces a small separation along the z-axis. By small here we mean the centre of each packet is separated by less than the width of each packet.

Finally the weakly separated packets enter another magnet, $\mathrm{SG_x}$, this time with its field oriented along the x-axis. Here a strong measurement takes place along the x-axis. Note that $\mathrm{SG_z}$ and $\mathrm{SG_x}$ are measuring conjugate variables.

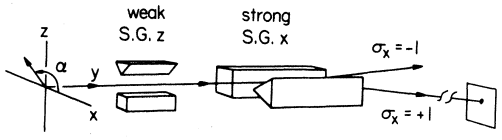

Figure 6.4 The modified SG apparatus to observe and measure the weak value for spin-1/2 particles. The diagram clearly shows the three stages of the weak measurement process; preparation, weak and strong (Duck *et al* 1989).

6.2.3 Details of the Weak Measurement of Spin.

Having outlined what the outcome of a weak measurement of spin will give us, let us now look into the details. We start, again, with the interaction Hamiltonian $H_I = g(t)\hat{Z}\hat{\sigma}_z$, which, as we have seen, gives a final state

$$|\psi(t)\rangle|\xi_f\rangle = e^{i\eta\hat{Z}\hat{\sigma}_z}|\psi(t_0)|\xi_i\rangle.$$

Now multiplying through by $\langle\xi_f|$ we find

$$|\psi(t)\rangle = \langle\xi_f|e^{i\eta\hat{Z}\hat{\sigma}_z}|\xi_i\rangle|\psi(t_0)\rangle$$

Expanding the expontional in the first term on the RHS of this equation, we find

$$\langle\xi_f|e^{i\eta\hat{Z}\hat{\sigma}_z}|\xi_i\rangle = \langle\xi_f|\xi_i\rangle\sum_{m=0}^{m=\infty}\frac{(i\eta\hat{Z})^m}{m!}\frac{\langle\xi_f|\hat{\sigma}_z^m|\xi_i\rangle}{\langle\xi_f|\xi_i\rangle}$$

We can now rewrite this expression as

$$\langle\xi_f|e^{i\eta\hat{Z}\hat{\sigma}_z}|\xi_i\rangle = \langle\xi_f|\xi_i\rangle\left[e^{i\eta\hat{Z}\langle\sigma_z\rangle_W} + \sum_{m=2}\frac{(i\eta\hat{Z})^m}{m!}\left[\langle(\sigma_z)^m\rangle_W - \langle\sigma_z\rangle_W^m\right]\right] \qquad (6.8)$$

where $\langle(\sigma_z)^m\rangle_W$ is the 'weak value' of the operator $(\hat{\sigma}_z)^m$ (See Aharonov *et al.* 1988). When the term involving the sum $\sum_{m=2}$ in equation (6.8) is small and can be neglected, we can write the final state at the position z in the form

$$\langle\xi_f|e^{i\eta\hat{Z}\hat{\sigma}_z}|\xi_i\rangle = \langle\xi_f|\xi_i\rangle e^{i\eta\hat{Z}\langle\sigma_z\rangle_W}$$

If we choose the initial position wave function to be a Gaussian so that

$$|\psi(t_0)\rangle = \int\exp\left[-\frac{z'^2}{4(\Delta z)^2}\right]|z'\rangle dz'$$

in the z-representation. Then we have

$$\psi(z,t) = e^{i\eta z\langle\sigma_z\rangle_W}\exp\left[-\frac{z^2}{4(\Delta z)^2}\right]$$

Recall that the weak value can be a complex number $\langle\sigma_z\rangle_W = \Re\langle\sigma_z\rangle_W + i\Im\langle\sigma_z\rangle_W$. This means that the imaginary part of the weak value shifts the real part of the the wave function to be centred around $z = 2(\Delta z)^2\Im\langle\sigma_z\rangle_W$ since, in this case the wave function is

$$\psi(z,t) \propto \exp\left[-\frac{(z + 2(\Delta z)^2\eta\Im\langle\sigma_z\rangle_W)^2}{4(\Delta z)^2}\right].$$

If we want to evaluate the real part of the weak value, we need to consider a wave function in the p-representation when we find

$$\phi(p,t) \propto \exp[-(\Delta z)^2(p - \eta\Re\langle\sigma_z\rangle_W)^2]. \tag{6.9}$$

In order to analyse the results further we must pass the beam through a Stern-Gerlach magnet aligned in the x-direction. This will split the beam into two components which will enable us to measure either

$$\langle\sigma_z\rangle_{W(+x)} = \frac{\langle +x|\hat{\sigma}_z|\xi_i\rangle}{\langle +x|\xi_i\rangle} \qquad \text{or} \qquad \langle\sigma_z\rangle_{W(-x)} = \frac{\langle -x|\hat{\sigma}_z|\xi_i\rangle}{\langle -x|\xi_i\rangle}$$

If the final screens shown in Figure 6.4 are placed sufficiently far from the last magnet, [strong SG x], so that the displacement in the x-direction will be greater than width of the wave packet, we will be able to measure the displacement of each beam from which we can determine the respective weak values.

Let us now compare the result given by equation (6.9) with the first term of equation (6.7), we see that the centroid of the Gaussian is shifted by an amount that depends upon the initial angle of polarisation α. Duck *et al* (1989) show that by a spefic choice of α, the difference can be greater by a factor of 70.

What we see from the above results is that although the spin part of the quantum state has collapsed, the spatial part does not collapse, it is shifted as a whole. The shift of the maximum gives us the weak value, $\langle\sigma_z\rangle_W$. This is a clear example of what Aharonov and Vaidman (1993), (1995), call a 'protected' wave function.

Of course in actually carrying out the experiment we have to ensure that the neglected terms in equation (6.8) are satisfied. Also care must be taken to ensure that the terms $\partial B_x/\partial x$ and $\partial B_y/\partial y$, which cannot vanish because of the condition $\nabla.\boldsymbol{B} = 0$ must be taken into account. Detailed discussions of these effects is presented in Aharonov, Albert and Vaidman (1988) and in Duck, Stevenson and Sudarshan (1989) and will not be discussed further here.

If all these conditions are satisfied, the theory predicts that the x-displacement is greater than expected, producing an amplification effect. If the separation of the spin-z-up and spin-z-down wave packets produced in a normal Stern-Gerlach is d, then the weak measurement process should separate them by up to a distance $70d$.

We see here the amplification generated by the weak measurement comes about.

Notice no external amplification process is being used, only a subtle manipulation of the wave function of the atom occurs. In a sense this could be viewed as *self-amplification* with the advantage that no noise is generated, unlike real life amplifiers where irreducible distortion of the signal is inevitable. As we go deeper and deeper into the atomic and sub-atomic world signals are getting smaller and smaller. Therefore the weak measurement technique could offer the possibility of a new way of making observations that are hidden by insufficient resolution and the noise of the measuring instruments.

6.2.4 Experimental Realisation of Weak Stern-Gerlach Measurement using Photons.

The first experiments to test these ideas have used optical analogues. For example Ritchie, Story and Hulet (1991) used a Gaussian-mode laser source and the Stern-Gerlach magnets were replaced by optical polarisers. The weak measurement was performed by a thin birefringent-crystalline quartz plate as shown in Figure 6.5. The analysis produces a new feature which must be taken into account.

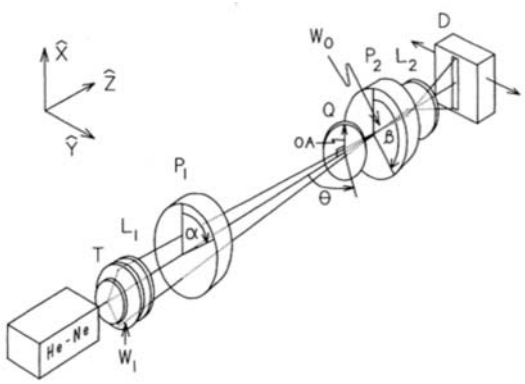

Figure 6.5 Weak Stern-Gerlach measurement using photons. Richie *et al* 1991. P_1 and P_2 are the polarisers, AO is the thin uniaxial birefringent crystal responsible for the weak effect.

The laser light travelling, in this case, along the z-axis is polarised at an angle α to the x-axis. The electric field of the input light is

$$\boldsymbol{E}_i = E_0 \exp\left[-\frac{x^2+y^2}{\Delta^2}\right](\cos\alpha\hat{\boldsymbol{x}} + \sin\alpha\hat{\boldsymbol{y}})$$

where Δ is the width of the beam. The birefingent plate is a plane-parallel uniaxial crystal whose optic axis is along the x-axis, so that the ordinary ray goes straight

through, while the extraordinary ray is deflected in the y-direction. As a result the two components corresponding to different polarisation states become separated by a distance a, which is made small compared with the overall beam width Δ. The emerging beam is then described by

$$\boldsymbol{E}_W = E_0 \exp\left[\frac{-x^2}{\Delta^2}\right]\left[\cos\alpha \exp\left[\frac{-(y+a)^2}{\Delta}\right] e^{i\phi}\hat{\boldsymbol{x}} + \sin\alpha \exp\left[\frac{-y^2}{\Delta^2}\right]\hat{\boldsymbol{y}}\right]$$

Here ϕ is the phase difference arising from the difference in the optical path lengths of the two rays.

The post selection is made using a polariser aligned at an angle β to the x-axis. The resulting wave function is

$$\boldsymbol{E}_f = E_0 \exp\left[\frac{-x^2}{\Delta^2}\right]\left[\cos\alpha\cos\beta \exp\left[\frac{-(y+a)^2}{\Delta}\right] e^{i\phi} + \sin\alpha\sin\beta \exp\left[\frac{-y^2}{\Delta^2}\right]\right]$$
$$\times(\cos\beta\hat{\boldsymbol{x}} + \sin\beta\hat{\boldsymbol{y}})$$

Thus we see in this example that there are two distinct effects produced by the uniaxial crystal. Firstly there is a deflection in the y-direction of one component of the beam relative to the other. Secondly there is a new feature, namely, a phase change between the two components due to the difference of the speeds in the two components in the crystal. In the magnetic analogue discussed earlier, only one of these effects is present, namely, a deflection between the two spin components. There is no phase factor change in the magnetic case.

If we choose $\alpha = \beta$, we find the two Gaussians add constructively to produce an intensity that is approximately an unshifted Gaussian. Suppose we rotate the second polariser to an angle $\beta = \alpha + \pi/2 + \epsilon$ with $\epsilon \ll 1$ and suppose $a/2\Delta \ll \epsilon \ll 1$, the interference will produce a single Gaussian with its centre shifted by the weak value $\langle S_y\rangle_W \approx a\cot(\epsilon)/2$. this is much larger then a. In the actual experiments an amplification of about 100 could be achieved.

To summarise, a weak measurement involves measuring a change of phase of the wave function produced by an operator rather then inducing a collapse of the wave function into an eigenfunction of that particular operator. But this does not mean that we have avoided a strong measurement altogether. The final stage of the measurement is a strong measurement, in the case we are discussing, it is a spin measurement in the x-direction. This final process collapses the *spin* part of the wave function, but leaves the form of the spatial part unchanged but displaced as a whole. It is from this displacement that the weak value can be found. Notice, to find the shape of the wave function we must perform a series of measurements on different particles, each being produced with the same initial wave function. The

fact that the result is obtained statistically is not a problem. A pair of numerical values can be given to the weak values.

Having illustrated how weak measurements can be realised, we now need to explain exactly how these weak values are related to the conventional approach (see Hosoya and Shikano (2010) and then to show how these weak values are related to the Bohm energy and momentum used in the Bohm model (see Bohm and Hiley (1993).

6.2.5 Weak Values.

Let us now examine the meaning of weak values defined by

$$\langle A(t)\rangle_W = \frac{\langle\phi(t)|\hat{A}|\psi(t_0)\rangle}{\langle\phi(t)|\psi(t_0)\rangle}$$

in more detail. From their very definition, it seems rather strange to call them 'values', as they are complex numbers. They are weighted transition probability amplitudes and we can use them to give us an expression for the probability amplitude of going from $|\psi\rangle$ to $|\phi\rangle$. This latter state is called the 'post-selected' state. Again the weighting function, $\langle\phi|\psi\rangle$, in the denominator is strange, since its magnitude depends on the relation between the pre- and post-selected states. This means that in some situations it can be chosen to make the weak value large by choosing the post-selected state $|\phi(t)\rangle$ to be nearly orthogonal to the initial state $|\psi(t_0)\rangle$, giving rise to the amplification discussed earlier.

To bring out the meaning of weak values more clearly, let us first note that the usual expression for the mean value of the Hermitean operator $\hat{A}$, sometimes known as a bilinear invariant, is a weighted sum of transition probability amplitudes $\langle\phi_j|\hat{A}|\psi\rangle$ and we can write

$$\langle\psi|\hat{A}|\psi\rangle = \sum_j \langle\psi|\phi_j\rangle\langle\phi_j|\hat{A}|\psi\rangle$$

where $|\phi_j\rangle$ is a complete set of orthonormal states.

However the mean value can also be written in the form (see Hosoya and Shikano 2010)

$$\langle\psi|\hat{A}|\psi\rangle = \sum \langle\psi|\phi_j\rangle\left(\frac{\langle\phi_j|\psi\rangle}{\langle\phi_j|\psi\rangle}\right)\langle\phi_j|\hat{A}|\psi\rangle = \sum \rho_j(\phi)\frac{\langle\phi_j|\hat{A}|\psi\rangle}{\langle\phi_j|\psi\rangle}. \tag{6.10}$$

Here $\rho_j(\phi)$ is the probability of finding the system in the state $|\phi_j\rangle$. Now we can see that if $|\phi_j\rangle$ and $|\psi\rangle$ are close to being orthogonal, the probability of finding a large weak value is small even though the 'weak value' itself may be large. In other words, large numerical values for the weak value make a correspondingly small

contribution to the overall mean value. The puzzling feature is why these weak values should be of any interest at all. Before going on to discuss this, let us make one more point.

To obtain a more comfortable feel for the weak values, notice the special case when we choose the $|\phi_j\rangle$ to be a set of eigenfunctions of the operator $\hat{A}$ i.e. $\hat{A}|a_j\rangle = a_j|a_j\rangle$, then we obtain the standard result

$$\langle\psi|\hat{A}|\psi\rangle = \sum \rho_j(a)a_j.$$

Thus the meaning of post-selection with a set of eigenstates of $\hat{A}$, namely $|a_j\rangle$, is very clear, but the meaning when a different set of states, $|\psi_j\rangle$, are post selected is still not clear. However certain weak values do have a specific meaning in the context of the Bohm model as we will now show, but first we must have a closer look at the meaning of bilinear invariants from a more general point of view.

6.3 Bilinear Invariants

6.3.1 Bilinear Invariants of the Second Kind.

So far we have confined our discussion to bilinear invariants of the form $\langle\psi|\hat{A}|\psi\rangle$. However Takabayashi (1955) points out that when the operators $\hat{A}$ are represented as derivatives, we need to introduce what he calls 'bilinear invariants of the second kind' in order to have a complete specification of a quantum system. These invariants can be written in the form

$$(\partial\psi)\psi^* \pm \psi(\partial\psi^*).$$

Here we have written a generic ∂ for operators like $\partial/\partial x,\ \partial^2/\partial x^2,\ \partial/\partial t \ldots$. If we take the plus sign, we will simply have a derivation, which will give us a bilinear invariant of the first kind. Taking the minus sign gives us a bilinear invariant of the second kind. We will follow convention and write this as

$$\psi \overleftrightarrow{\partial} \psi^* = (\partial\psi)\psi^* - \psi(\partial\psi^*).$$

To understand the meaning of these invariants we need to return to consider the energy-momentum tensor.

6.3.2 The Energy-momentum Tensor

As is well known, the Schrödinger equation can be derived from the Langrangian

$$\mathcal{L} = -\frac{1}{2m}\nabla\psi^* \cdot \nabla\psi + \frac{i}{2}[(\partial_t\psi)\psi^* - (\partial_t\psi^*)\psi] - V\psi^*\psi.$$

The details will be found in Heisenberg (1949). Using the energy-momentum tensor defined by

$$T^{\mu\nu} = -\left\{\frac{\partial\mathcal{L}}{\partial(\partial^\mu\psi)}\partial^\nu\psi + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\psi^*)}\partial^\nu\psi^*\right\},$$

we find the momentum density can be written as

$$T^{0j} = -\left\{\frac{\partial\mathcal{L}}{\partial(\partial^0\psi)}\partial^j\psi + \frac{\partial\mathcal{L}}{\partial(\partial^0\psi^*)}\partial^j\psi^*\right\}.$$

Because

$$\frac{\partial\mathcal{L}}{\partial(\partial^0\psi)} = \frac{1}{2i}\psi^* \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial(\partial^0\psi^*)} = -\frac{1}{2i}\psi,$$

we find

$$T^{0j} = \frac{i}{2}\left[\psi^*\partial^j\psi - \psi\partial^j\psi^*\right].$$

we see immediately that this is the expression for the current, as expected. If we write $\partial^j = -\nabla$, we immediately recognise that $T^{0j}(\boldsymbol{x}, t)$ is the Bohm momentum, $\boldsymbol{p}_B(\boldsymbol{x}, t) = \boldsymbol{\nabla}S(\boldsymbol{x}, t)$, or the so called 'guidance condition', as can be easily seen by writing $\psi = R\exp[iS]$. A similar argument using

$$T^{00} = \frac{i}{2}[\psi^*\partial^0\psi - \psi\partial^0\psi^*].$$

gives the Bohm energy, $E_B(\boldsymbol{x}, t) = -\partial_t S(\boldsymbol{x}, t)$. Thus the Bohm energy and momentum are nothing but the energy-momentum density derived from the standard expression for the energy-momentum tensor. Hiley and Callaghan (2010C) have shown that similar relationships hold for the Pauli and Dirac particles.

In the standard approach the energy, E, is defined as

$$E(t) = \int T^{00}(\boldsymbol{x}, t)d^3x \tag{6.11}$$

while the momentum P^j is defined by

$$P^j(t) = \int T^{0j}(\boldsymbol{x}, t)d^3x. \tag{6.12}$$

These quantities are not a function of position but are global quantities. Thus the conventional theory is concerned with the *global* energy and momentum, whereas the Bohm approach focuses on the *local* expressions, $E_B(\boldsymbol{x}, t), P^j_B(\boldsymbol{x}, t)$. Conservation of global energy and momentum is achieved through

$$\frac{d}{dt}\int T^{0\mu}(\boldsymbol{x}, t)d^3x = 0 \qquad \mu = 0, 1, 2, 3.$$

As we have already pointed out in section 1, conservation of local energy is achieved using the quantum Hamilton-Jacobi equation (6.2) (seevHiley and Callaghan 2010C). The differences between these two approaches to the notion of particle number has been discussed by Colosi and Rovelli (2009).

6.3.3 Weak Values and the $T^{0\mu}(\boldsymbol{x},t)$ components of the Energy-momentum Tensor.

We will now show how the $T^{0\mu}(\boldsymbol{x},t)$ components of the energy-momentum tensor are related to weak values. To do this we take equation (6.10) and replace the operator $\hat{A}$ with the momentum operator $\widehat{P}_x$, while replacing $\langle\phi|$ by $\langle x|$. This means we are investigating the weak value of the momentum at a post selected position x. Then

$$\langle\psi(t)|\widehat{P}_x|\psi(t)\rangle = \int \rho(x,t)\frac{\langle x|\widehat{P}_x|\psi(t)\rangle}{\langle x|\psi(t)\rangle}dx.$$

Now we can write

$$\langle x|\widehat{P}_x|\psi(t)\rangle = \int \langle x|\widehat{P}_x|x'\rangle\langle x'|\psi(t)\rangle dx'.$$

However we use $\langle x|\widehat{P}_x|x'\rangle = -i\nabla_x\delta(x-x')$ again, so that

$$\langle x|\widehat{P}_x|\psi(t)\rangle = -i\nabla_x\psi(x,t).$$

If we now write $\psi(x,t) = R(x,t)e^{iS(x,t)}$ we find

$$\frac{\langle x|\widehat{P}_x|\psi(t)\rangle}{\langle x|\psi(t)\rangle} = \nabla_x S(x,t) - i\nabla_x\rho(x,t)/2\rho(x,t)$$

where $\rho(x,t) = |\psi(x,t)|^2$ is the usual probability density. If we now form a bilinear invariant of the second kind by writing

$$\frac{\langle x|\overleftrightarrow{P}_x|\psi(t)\rangle}{\langle x|\psi(t)\rangle} = \nabla_x S(x,t). \tag{6.13}$$

We see that this invariant is, in fact, the $T^{01}(\boldsymbol{x},t)$ component of the energy-momentum tensor which also happens to be the x-component of the Bohm momentum (Leavens, 2005 and Howard 2007). The imaginary part of the weak value is the osmotic velocity which is generally ignored, but see Bohm and Hiley (1989) for further details.

If we introduce formally the time operator $\widehat{P}_t$, the real part of the weak value $\langle P_t\rangle_W$ gives $T^{00}(\boldsymbol{x},t)$ and hence the Bohm energy, while the real part of $\langle P_t^2\rangle_W$ gives the Bohm kinetic energy plus the quantum potential (Leavens 2005). The corresponding expressions for a particle with spin has been presented in Hiley (2012).

Having seen the relevance of weak values, we now need to show how they can be realised experimentally.

6.4 Weak Measurements with Photons.

6.4.1 The Experiment of Kocis et al.

Weak measurements have already been made on very weak photon beams in a beautiful experiment by Kocsis *et al.* 2011) (See Figure 6.1). Polarised single photons are split by a 50-50 beam splitter before being re-coupled using two collimated fibre couplers that act as two slits. After the photon had passed through the screen containing the two slits, the real part of the weak value of transverse momentum is measured at various points. From these values a set of stream lines are constructed which have a striking similarity to the trajectories calculated by Philippidis, Dewdney and Hiley (1979) for a Schrödinger particle described by the Bohm model (See Figure 6.2).

The weak measurement of the momentum was made by introducing a thin slither of birefringent calcite crystal into the photon beam immediately after the two-slits. The calcite was thin enough and placed with its optical axis suitably oriented so that the ordinary and extraordinary ray were still over-lapping after they left the crystal. This process induces a phase change in the beam that can be calculated using the interaction Hamiltonian $\widehat{H}_I = g\widehat{P}_x\widehat{S}_1$, where $2\widehat{S}_1 = [|H\rangle\langle H| - |V\rangle\langle V|]$. Here $|H\rangle$ and $|V\rangle$) are the horizontal and vertical components of the polarisation of the photon. This gaves the final state of the photons to first order to be

$$\Psi(x_f) \approx \frac{\psi(x_f)}{\sqrt{2}}\left[e^{-i\phi}|H\rangle + e^{i\phi}|V\rangle\right]$$

where we have written $\phi = \frac{D\Delta t}{2}\langle P_x\rangle_W$, the theoretical value of the phase. In the Kocsis *et al* experiment, the phase was written as $\phi = \zeta\langle P_x\rangle_W$ where ζ is some factor that depends on the details of the experimental set up as discussed in Kocsis *et al.* (2011a).

To measure this phase factor, we need to perform a strong measurement using $S_3 = |R\rangle\langle R| - |L\rangle\langle L|$. Here $|R\rangle$ and $|L\rangle$ are the right and left circular polarised states respectively. Then we find that

$$\langle S_3\rangle = \sin[\zeta\langle P_x\rangle_W].$$

The weak value of the transverse momentum can then be found from the difference in the counts of the number of right hand photons and the left hand photons arriving

at a point through the relation

$$\langle P_x \rangle_W = \frac{1}{\zeta} \sin^{-1}\left(\frac{I_R - I_L}{I_R + I_L}\right).$$

This then determines the transverse momentum at a series of points in the interference region. From these values, together with the momentum along the axis, Kocsis *et al.* (2011) construct the momentum stream lines as explained in detail in their paper.

6.4.2 The Meaning of the Stream Lines.

The question that must now be addressed is precisely what meaning can be attached to these streamlines in the case of photons. Since we are here dealing with the electromagnetic field, a weak measurement of the momentum will give values of the Poynting vector of this field at various points after the field has passed through the thin calcite crystal used to perform the weak measurement. The experiment of Kocsis *et al.* (2011) is dealing with single photons posing the question "What is the meaning of the Poynting vector when a single photon is involved?"

To make the discussion more transparent, let us consider first a measurement that would determine the $T^{00}(\boldsymbol{x}.t)$ component, namely, the electromagnetic energy. If the beam was a monochromatic plane wave composed of many photons, we would argue, along with Dirac (1927), that if we divide this value by the number of photons, we could determine the energy carried by a single photon.

This type of argument has already been used in the Beth (1936) experiment to measure mechanically the spin of a photon. In this experiment a circularly polarised beam of light is reflected back by a mirror after passing through a quartz wave plate. If the quartz wave plate is fixed to the mirror and the combined system attached to a quartz fibre, the change of the beams angular momentum in the wave plate at reflection will produce a torque in this fibre. The change of angular momentum taking place in the quarter-wave plate induces a torque on the quartz wave plate causing it to rotate through an angle that enables the change in angular momentum of the reflected beam to be measured. When one divides this value by the number of photons in the beam, we find that each photon carries a unit of angular momentum as expected.

Since in the Kocsis *et al* experiment, only a single photon enters the apparatus at a time, the weak measurement of the energy will give a value for the energy carried by a single photon. A corresponding measurement of the momentum will give us the momentum of a single photon.

We are now faced with an interesting question, "What is the meaning of the energy and momentum of a photon at a point?" The photon has a sharply defined

energy and momentum so it could be argued that the uncertainty principle would forbid us the talk about a photon at a sharply defined position. Detailed discussions of these problems can be found in Cook (1982), Mandel (1983) and Roychoudhuri, Kracklauer and Creath (2008).

In the Bohm approach, sharp 'values' can be simultaneously attributed to an object (the beables), but if we try to attribute a simultaneous position and momentum to a photon and while maintaining the field values for the energy, $T^{00} = (E^2 + B^2)/8\pi$, and momentum, $T^{0i} = (\boldsymbol{B} \times \boldsymbol{E})^i/4\pi$, we run into trouble. This problem has already been pointed out by Bohm, Hiley and Kaloyerou (1987) and by Bohm and Hiley (1993), but its significance does not seem to have been appreciated so we will briefly outline the problem using a simple example.

Consider the special case in which $\boldsymbol{B}$ is parallel to $\boldsymbol{E}$ so that $\boldsymbol{E} \times \boldsymbol{B} = 0$. Assume further $\boldsymbol{E}$ and $\boldsymbol{B}$ are in the x^1 direction. This means that under Lorentz boost in the x^1 direction, we will still have $\boldsymbol{E}'$ parallel to $\boldsymbol{B}'$ so that $(\boldsymbol{E}' \times \boldsymbol{B}') = 0$. It follows that the velocity of the photon is zero in a whole range of Lorentz frames, which makes little sense. Indeed no meaning can be given to a photon at rest in general. This gives rise to serious doubts as to whether one can think of the photon as a particle even in the Bohmian sense.

Indeed this was one of the reasons why Bohm (1952b), Bohm, Hiley and Kaloyerou (1987) and Bohm and Hiley (1993) proposed that photons, and bosons in general, should be treated in terms of a field. This meant developing a Bohmian approach to quantum field theories (Bohm *et al.*, 1987). This theory treats the field $\phi(x^\mu)$ and its conjugate momentum $\pi(x^\mu)$ as the beables. The dynamics is introduced in terms of a super-Schrödinger equation, which, in turn, leads to a super-quantum Hamilton-Jacobi equation together with a super-quantum potential which organises the fields to produce the quantum effects. We will not discuss this approach here but the details for a scalar field can be found in Bohm, Hiley and Kaloyerou (1987) while those for the electromagnetic field can be found in Kaloyerou (1994). A simplified account can also be found in Bohm and Hiley (1993).

For our purposes here, all we need to know is how the concept of a photon arises in Bohm's approach to the field. When an atom emits a quantum of energy, the energy disperses by spreading into the global electromagnetic field so that the energy of the field increases by $h\nu$. This energy is not localised and this is reflected in standard quantum field theory where an integral over all space is performed as shown in equations (6.11) and (6.12).

The atom absorbs energy from the field through the action of the super-quantum potential. This potential is non-local and sweeps up a quantum of energy, $h\nu$, from the field, causing an appropriate local transition in the atom. It is the structure of the energy levels in the atom that determines the amount of energy absorbed or emitted.

More details can be found in Bohm, Hiley and Kaloyerou (1987) and Kaloyerou (1994).

This is a radically different way of understanding the concept of a photon from the usual one, although even in more conventional case there is no clear view as to how we are to understand the physical nature of the photon (See Roychoudhuri, Kracklauer and Creath 2008). Not only does it avoid the difficulties we have discussed above, but it also offers an explanation for the coherent state where, although the quantum state of the field is well defined, the number of photons it contains is not. If photons existed as well defined actual entities then it is hard to understand why the number of entities in a definite state should be ambiguous. A further advantage to this approach is that the non-local nature of the quantum potential gives an account of the EPR effect as was shown in detail in Bohm *et al.*, (1987)

Returning to the photons considered in the experiment of Kocsis *et al*, we note that the individual photons are described by a *Gaussian wave packet*, indicating that the photons are not mono-energetic, a small spread of frequencies is involved. In order to find the weak value of the momentum at any point, we must place a detector at that point and it is by counting photons that the weak values are determined. So in terms of this model, the stream-lines that Kocis *et al* (2011) construct are simply energy-momentum flow lines and do not imply that photons actually follow these flow lines.

6.4.3 Schrödinger Particle Trajectories.

The objections that we have raised against photon trajectories does not apply to atoms. If we assume that atoms can be described by the Schrödinger equation, then, as we have already pointed out, the Bohm approach allows us to calculate trajectories in a two-slit interference experiment as was shown by Philippidis *et al* (1979).

An atom, with its finite rest mass, can be brought to rest and can thus have a simultaneous position and momentum, even though we will not be able to measure these values simultaneously using a von Neumann-type strong measurement. However as we have seen above in section 6.3.3, this simultaneous momentum is the Bohm momentum, which is not an eigenvalue of the momentum operator in the state under consideration, but the weak value defined in equation (6.13). This means that this momentum can be measured by making the appropriate weak measurement of the momentum operator. Since atoms move relatively slowly, the question of Lorentz invariance will not arise so that we can discuss the notion of a particle trajectory without encountering the problems that beset the notion of a photon trajectory. Thus in this case a measurement of the appropriate weak values will

then allow us to construct an ensemble of trajectories which can then be compared with the theoretical calculations of Philippidis *et al.* (1979).

6.5 Conclusions.

In this paper we have discussed how weak measurements give new information about quantum processes. In particular we have shown how weak values give new information about spin and about energy and momentum. In fact we have shown that a weak value of the energy and momentum operators using position as a post selection gives us values for the $\{0, \mu\}$ components of the energy-momentum tensor $T^{\mu\nu}(\boldsymbol{x}, t)$. In turn we show that the Bohm energy, $E_B(\boldsymbol{x}, t)$, and the Bohm momentum, $P_B(\boldsymbol{x}, t)$ are simply related to the appropriate values of $T^{0\mu}(\boldsymbol{x}, t)$. Thus values for these Bohm variables can be obtained by the measurement of the corresponding weak values, showing that the Bohm approach is not a return to classical notions. It gives us information about the local energy and momentum whereas the standard interpretation gives us information about their global properties.

To date the weak measurement techniques described in this paper has been applied only to optical systems such as the Young's two slit experiment (Kocsis *et al.* 2011) and the optical analogue of the classic Stern-Gerlach experiment (Ritchie *et al.* 1991). What has not yet been demonstrated is the use of weak measurements for particles with non-zero rest mass obeying the Schrödinger equation. Results here will enable us to make a direct comparison with the theoretical predictions of the Bohm model.

With this objective in mind we propose to carry out a series of experiments starting with a modified SG apparatus shown in Figure 6.4. We plan to confirm that the amplifications factors that are measured correspond to those predicted. We then plan to apply the method to a modified Young's slits apparatus aiming to reproduce the 'photon trajectories' found by Kocsis *et al.* (2011), only in our case we will use atoms. This will open the possibility of employing the technique to observe weak effects assumed to be beyond our reach using conventional techniques. In this way we believe that we can make experiments that observe quantum effects that were once thought to be impossible.

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6.7 References

Aharonov, Y., Albert, D. Z. and Vaidman, L. (1988) How the Result of a Measurement of a Component of the Spin of a Spin-1/2 Particle Can Turn Out to be 100, *Phys. Rev. Lett.*, **60**, 1351-4.

Aharonov, Y. and Vaidman, L., (1990) Properties of a quantum system during the time interval between measurements, *Phys. Rev. A*, **41**, 11-19.

Aharonov, Y. and Vaidman, L., (1993) Measurement of the Schrödinger wave of a single particle, *Phys. Lett.* **A178**, 38-42.

Aharonov, Y. and Vaidman, L. (1995), Protective Measurements. *Annals of the New York Academy of Sciences*, **755**, 361373.
doi: 10.1111/j.1749-6632.1995.tb38979.x

Beth, R. A., (1936) Mechanical Detection and Measurement of the Angular Momentum of Light, *Phys. Rev.*, **50**, 115-125

Bohm, D., (1951) *Quantum Theory*, Prentice-Hall, Englewood Cliffs, N.J.

Bohm, D., (1952) A Suggested Interpretation of the Quantum Theory in Terms of Hidden Variables, I, *Phys. Rev*., **85**,166-179.

Bohm, D., (1952) A Suggested Interpretation of the Quantum Theory in Terms of Hidden Variables, II, *Phys. Rev*., **85**, 180-193.

Bohm, D., Hiley, B. J. and Kaloyerou, P.N., (1987) An Ontological Basis for the Quantum Theory: II -A Causal Interpretation of Quantum Fields, *Phys. Reports*, **144**, 349-375.

Bohm, D. and Hiley, B. J., (1989) Non-locality and Locality in the Stochastic Interpretation of Quantum Mechanics, *Phys. Reps*, **172**, 92-122.

Bohm, D. and Hiley, B. J., (1993) The Undivided Universe: an Ontological Interpretation of Quantum Theory, Routledge, London.

Colosi, D., and Rovelli, C., (2009) What is a particle?. *Classical and Quantum Gravity*, **26**, 025002.

Cook, R. J., (1982) Photon Dynamics, *Phys. Rev. A,* **25**, 2164-67.

Dirac, P. A. M., (1927) The Quantum Theory of the Emission and Absorption of Radiation, *Proc. Roy. Soc.* **114A**, 243-265.

Duck, I. M., Stevenson, P. M. and Sudarshan, E. C. G., (1989) The sense in which a "weak measurement" of a spin-1/2 particle's spin component yields a value 100, *Phys. Rev. D*, **40**, 2112-17.

Dürr, D., Goldstein, S. and Zanghi, N., (1996) Bohmian Mechanics as the Foundation of Quantum Mechanics, in *Bohmian Mechanics and Quantum Theory:*

an Appraisal, ed J. T. Cushing, A Fine and S. Goldstein, Boston Studies in the Philosophy of Science, **184**, 21-44, Kluwer, Dordrecht.

Gerlach, W. and Stern, O., (1922) Der experimentelle Nachweiss der Richtungsquantelung im Magnetfeld, *Zeits.Phys.*, **9**, 349-355.

Heisenberg, W., (1949) *The Physical Principles of Quantum Mechanics,* Trans. by C. Eckart and F. C. Hoyt, Dover, New York.

Heisenberg, W., (1958) *Physics and Philosophy: the revolution in modern science*, George Allen and Unwin , London.

Hiley, B. J., and Callaghan, R. E., (2010a) The Clifford Algebra approach to Quantum Mechanics A: The Schrödinger and Pauli Particles. arXiv:Maths-ph: 1011.4031.

Hiley, B. J., and Callaghan, R. E., (2010b) The Clifford Algebra Approach to Quantum Mechanics B: The Dirac Particle and its relation to the Bohm Approach. arXiv: Maths-ph:1011.4033.

Hiley, B. J., Callaghan, R. E., (2011c) Clifford Algebras and the Dirac-Bohm Quantum Hamilton-Jacobi Equation, *Foundations of Physics*, **42**, 192-208. DOI: 10.1007/s10701-011-9558-z

Hiley, B. J., (2012) Weak Values:Approach through the Clifford and Moyal Algebras, *J. Phys.: Conference Series*, **361**, 012014. Doi: 10.1088/1742-6596/361/1/012014.

Hosoya, A. and Shikano, Y., (2010) Strange Weak Values, *J. Phys. A: Math. Theor.*. **43**, 385307.

Kaloyerou, P. N., (1994) The Causal Interpretation of the Electromagnetic field, *Phys. Rep*., **244**, 287-358.

Kocis, S., Braverman, B., Ravets, S., Stevens, M. J., Mirin, R. P., Shalm, L.K., Steinberg, M. A., (2011) Observing the Average Trajectories of Single Photons in a Two-Slit Interferometer, *Science*. **332**, 1170-73.

Kocis, S., Braverman, B., Ravets, S., Stevens, M. J., Mirin, R. P., Shalm, L.K., Steinberg, M. A., (2011a) *ibid*, Supporting online material, www.sciencemag.org/cgi/content/full/332/6034/1170/DC1.

Leavens, C. R., (2005) Weak Measurements from the point of view of Bohmian Mechanics, *Found. Phys.*, **35**, 469-91 doi: 10.1007/s10701-004-1984-8

Mandel, L., (1983) Photon interference and correlation effects produced by independent quantum sources, *Phys. Rev. A*, **28**, 929-43.

Philippidis, C., Dewdney, D., and Hiley, B. J., (1979) Quantum Interference and the Quantum Potential, *Nuovo Cimento* **52B**, 15-28.

Ritchie, N.W., Story, J. G., and Hulet, R. G., (1991) Realization of a Measurement of a Weak Value, *Phys. Rev. Lett.* **66**, 1107-10.

Roychoudhuri, C., Kracklauer, A. F., and Creath, K., (2008) eds. *The nature of light: what is a photon?*. CRC Press.

Schweber, S. S., (1961) *An Introduction to Relativistic Quantum Field theory*, Harper-Row, New York.

Takabayasi, T., (1955) On the Structure of Dirac Wave Function, *Progress of Theoretical Physics*, **13**, 106-108.

von Neumann, J., (1955) *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, Princeton.

Wiseman, H. M., (2007) Grounding Bohmian mechanics in weak values and Bayesianism, *New J. Phys*., **9**, 165-77. doi: 10.1088/1367-2630/9/6/165